We've previously seen constructions like products, equalizers, pullbacks, and also their dual counterparts. We have observed that every object had the following properties in their definitions:

1. A collection of objects and morphisms with which the objects are related through special morphisms,
2. A commutativity condition on the total collection of objects and morphisms,
3. And a finality condition over any other object with morphisms satisfying the above two.

This was not a mistake. These objects are examples of what are known as limits (and their duals colimits) and are crucial in the discussion both leading up to, and during defining limits. As a preliminary, we make the following definitions.

Definition 1. Let C be a category, and J a small category. A functor $\mathrm{F} : \mathrm{J} \to \mathrm{C}$ is called a J-shaped diagram, or a diagram of shape J, or simply, a diagram.

Examples of diagrams are plentiful, so one is encouraged to find some themselves. As a bit of foreshadowing, one can consider diagrams indexed by categories spanned by sets, diagrams of shape $\bullet \rightrightarrows \bullet$ and of shape $\bullet \to \bullet \leftarrow \bullet$ (and also their dual diagrams!)

Remark 1. Given a shape $\mathrm{J} \to \mathrm{C}$ and an object $c \in \mathrm{C}$, one can consider the *constant diagram of shape* J. This diagram, denoted $\Delta(c) : \mathrm{J} \to \mathrm{C}$ takes every object $j \in \mathrm{J}$ to the object $c \in \mathrm{C}$, and every morphism $\alpha : j \to k \in \mathrm{J}$ to the constant morphism $1_c : c \to c \in \mathrm{C}$. Note that we then have a functor called the "constant diagram functor" $\Delta : \mathrm{C} \to \mathrm{C}^{\mathrm{J}}$, sending every object c of C to its constant diagram $\Delta(c)$ and every morphism $f : c \to c'$ to the transformation sending every image to c' through post-composition. What we describe is:

$$\Delta(c) : \mathrm{J} \to \mathrm{C},\ \Delta(c)(j) := c,\ \Delta(c)(j \to k) := 1_c$$
$$\Delta(f : c \to c') : \Delta(c) \to \Delta(c'),\ \Delta(f)_j = f : c \to c'$$

$$\begin{tikzcd}
\Delta(c)_j \arrow[r, "\Delta(f)_j"] \arrow[d, equal] & \Delta(c')_j \arrow[d, equal] \\
c \arrow[r, "f"] & c'
\end{tikzcd}$$

Definition 2. Let C, D be categories and $\mathrm{F}, \mathrm{G} : \mathrm{C} \to \mathrm{D}$ functors between them. A natural transformation $\alpha : \mathrm{F} \Rightarrow \mathrm{G}$ is a collection of morphisms $(\alpha_c : \mathrm{F}c \to \mathrm{G}c)_{c \in \mathrm{C}}$ in D, indexed by the objects of C such that given any pair of objects $c, c' \in \mathrm{C}$, and any morphism $f : c \to c'$, all diagrams of the following shape commute:

$$\begin{tikzcd}
\mathrm{F}c \arrow[r, "\alpha_c"] \arrow[d, "\mathrm{F}f"'] & \mathrm{G}c \arrow[d, "\mathrm{G}f"] \\
\mathrm{F}c' \arrow[r, "\alpha_{c'}"'] & \mathrm{G}c'
\end{tikzcd}$$

One does not need to search much for such natural transformations. The concept of being natural in classical mathematics often refers to the idea that the morphism in question assembles into a natural transformation under the objects of the category. Some examples include:

- Suppose we have two chain complexes $C_\bullet$ and $D_\bullet$. A chain map between them is a collection of group homomorphisms such that the associated squares are commutative. Diagrammatically,

$$\begin{tikzcd} \dots \arrow[r] & C_n \arrow[r] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \dots \\ \dots \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & D_{n-2} \arrow[r] & \dots \end{tikzcd}$$

  commutes at each square. Now, if we were to consider such complexes as instead diagrams of shape $(\mathbb{N}, \leq)$ (the shape here does not matter, pick whatever linear poset that suits your homology theory), then the chain map $f : C_\bullet \to D_\bullet$ can instead be simplified to be a natural transformation $f : C_\bullet \Rightarrow D_\bullet$, with components $f_n : C_n \to D_n$.

- Let X be a set. We have the powerset functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$ taking each set to its powerset, and each morphism $f : X \to Y$ to its direct image morphism $(A \subset X) \mapsto (f(A) \subset Y)$. Now, additionally, there is also a transformation $X \hookrightarrow \mathscr{P}(X)$ taking each element x of X to its singleton set $\{x\}$, and each morphism $f : X \to Y$ to the morphism across commuting the associated diagram. It can be seen from the diagram that the naturality condition gives that $\mathscr{P}(f)(\{x\}) = \{f(x)\}$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & \mathscr{P}(X) \arrow[d, "\mathscr{P}(f)"] \\ Y \arrow[r, hook] & \mathscr{P}(Y) \end{tikzcd}$$

  The maps $X \hookrightarrow \mathscr{P}(X)$ can be shown to be natural in X, giving a natural transformation $1_{\mathsf{Set}} \Rightarrow \mathscr{P}$.

Returning to our previous constructions, we wish to generalize their definitional pattern as much as possible. The third condition is already in a categorical state, but the first two conditions may be generalized, which is what we will do now.

Definition 3 (Cone I). Let $F : J \to C$ be a diagram. a cone over F (or simply a cone of F) is an object $c \in C$, together with a collection of morphisms $(\lambda_j : c \to Fj)_{j \in J}$ such that given any pair of objects $j, k \in J$ and any morphism $\alpha : j \to k \in J$, the diagram in Figure 1 commutes.

Similarly, a cone under F (or simply a *cocone* of F) is an object $c' \in C$, along with a collection of morphisms $(\varepsilon_j : Fj \to c')_{j \in J} \in C$ such that given any pair of objects and morphisms from J the diagram in Figure 2 commutes.

$$\begin{tikzcd}
 & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\
Fj \arrow[rr, "F\alpha"'] & & Fk
\end{tikzcd}$$

Figure 1: Cone diagram

$$\begin{tikzcd}
Fj \arrow[rr, "F\alpha"] \arrow[dr, "\varepsilon_j"'] & & Fk \arrow[dl, "\varepsilon_k"] \\
 & c' &
\end{tikzcd}$$

Figure 2: Cocone diagram

Note that this definition captures the idea we had before with products, equalizers, pullbacks (even terminal objects!) in the sense that each such object we described had morphisms going down into the diagram. The commutativity condition is further induced by some morphisms that we hadn't seen, two of which are shown below.

$$\begin{tikzcd}
 & \text{equalizer} \arrow[dl, "\lambda_\bullet"'] \arrow[dr, dashed, "\lambda_{\text{commute}}"] & \\
\bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{pullback} \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, dashed, "\lambda_{\text{commute}}"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

This definition is visually intuitive, and certainly in-line with what we've seen so far, but the following alternative definition provides a cleaner description of the topic:

Definition 4 (Cone II)**.** Let $F : J \to C$ be a diagram. A cone over F with apex c is an object c along with a natural transformation $c \Rightarrow F$, where $c : J \to C$ is the constant diagram defined above. Dually, a cone under F with nadir c (or a cocone of F) is an object c along with a natural transformation $F \Rightarrow c$.

Exercise 1. Show that the two definitions of cones and cocones agree. As a hint, note that you can rewrite the first commutative diagram as

$$\begin{tikzcd}
 & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\
Fj \arrow[rr, "F\alpha"'] & & Fk
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
c \arrow[r, "1_c"] \arrow[d, "\lambda_j"'] & c \arrow[d, "\lambda_k"] \\
Fj \arrow[r, "F\alpha"'] & Fk
\end{tikzcd}$$

Fixing a specific diagram $F : J \to C$, we can talk about the category Cone(F) of cones over said diagram. Such a category has the cones and natural transformations as objects, and as morphisms: $f : c \to c'$ in C such that for any leg $\lambda_j : c \to Fj$ and

$\varepsilon_j : c' \to \mathrm{F}j$, the diagram below commutes

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & c' \arrow[dl, "\varepsilon_j"] \\ & \mathrm{F}j & \end{tikzcd}$$

which, in turn, means that $f^*(\Delta(c')) = \Delta(c)$ is also a cone over F. The cocone category of F is defined similarly. With this, we are ready to discuss limits and colimits.

Definition 5. Let $\mathrm{F} : \mathrm{J} \to \mathrm{C}$ be a diagram. A limit of the diagram F, denoted $\lim_{\mathrm{J}} \mathrm{F}$, is a terminal object in Cone(F). Dually, a colimit of the diagram, denoted $\mathrm{colim}_{\mathrm{J}} \mathrm{F}$ is an initial object in Cocone(F).

Theorem 1. *Given a diagram* $\mathrm{F} : \mathrm{J} \to \mathrm{C}$*, if the limit of the diagram exists, then it is unique up to isomorphism. Dually, the colimit is unique up to isomorphism if it exists.*

Proof. We prove the limit case. If the limit exists, then it is a terminal object of Cone(F). We know that any two terminal objects of a category are isomorphic. □

Some immediate examples are as follows:

(i) Let X be a set, and $\mathrm{J} = \mathbf{C}(\mathrm{X})$ the category generated by the set X. Then the object $\lim_{\mathrm{J}} \mathrm{F}$ is the product $\prod_{j \in \mathrm{J}} \mathrm{F}j$. Similarly, the coproduct $\coprod_{\mathrm{J}} \mathrm{F}j$ is the colimit of the same diagram.

(ii) Let J be the cospan category $\bullet \to \bullet \leftarrow \bullet$. The limit of this diagram is a pullback. In a dual fashion, the colimit of the span category $\bullet \leftarrow \bullet \to \bullet$ is a pushout. Notice that the arrows of both the cone and the diagram have inverted (as one would expect from the dual notion).

(iii) Consider the special case for the category $\mathrm{J} = \mathbf{C}(\emptyset)$. In this case, the limit of this diagram is just a terminal object in C, and the colimit is an initial object. This implies that a 0-ary product is just a terminal object, and dually for the initial object.

(iv) For the diagram of shape $\bullet \rightrightarrows \bullet$, we get the equalizer of the diagram. The dual notion gives the coequalizer of the dual diagram J^{op}.

(v) Here is one nontrivial example of a colimit, motivated by topology. Consider the construction of a CW complex with cells and attaching maps; the process is recursive, starting with a collection of vertices, and attaching cells along the boundaries we identify with using said attaching maps. The process for attaching each cell can be given by

$$\mathrm{X}_{n-1} \cup_{\varphi_{n-1,k}} \mathrm{D}^n$$

Note that this construction requires that the boundary of the cell we're attaching be identified, i.e., that the following diagram commutes for each attaching map $\varphi_{n-1,k}$.

$$\begin{tikzcd}
X_{n-1} \cup_{\varphi_{n-1,k}} D^n & D^n \arrow[l] \\
X_{n-1} \arrow[u] & S^{n-1} \arrow[l, "\varphi_{n-1,k}"] \arrow[u, hook]
\end{tikzcd}$$

One can see that this construction is a pushout along the attaching maps and the boundary inclusion maps.

Now, we can collect each such attaching map $\varphi_{n-1,k}$ in a collection J_{n-1}. Giving this set the discrete topology, we can instead say that the following diagram is a pushout diagram

$$\begin{tikzcd}
X_{n-1} \cup_{\overline{\varphi}_{n-1}} D^n & D^n \arrow[l] \\
X_{n-1} \arrow[u, "\gamma_{n-1}"] & J_{n-1} \times S^{n-1} \arrow[l, "\overline{\varphi}_{n-1}"] \arrow[u, hook]
\end{tikzcd}$$

where $\overline{\varphi}_{n-1}$ identifies each point $(\varphi_{n-1,k}, z) \mapsto \varphi_{n-1,k}(z)$. In this construction, we then set the n-skeleton to be $X_n := X_{n-1} \cup_{\overline{\varphi}_{n-1}} D^n$. Now, specifying the associated attaching maps, we obtain the following diagram, generated by the skeleta X_k and their relevant inclusion maps γ_k, generated by the above process:

$$X_0 \xrightarrow{\gamma_0} X_1 \xrightarrow{\gamma_1} X_2 \xrightarrow{\gamma_2} \dots$$

Then, the CW complex X is the smallest space containing each skeleton and the inclusions, or rather in categorical terms, $X := \mathrm{colim}_{n\in(\mathbb{N},\leq)} X_n$. Diagrammatically,

$$\begin{tikzcd}
 & & \mathrm{colim}_{n\in(\mathbb{N},\leq)} X_n & & \\
X_0 \arrow[urr, hook] \arrow[r, "\gamma_0"] & X_1 \arrow[ur, hook] \arrow[rr, "\gamma_1"] & & X_2 \arrow[ul, hook] \arrow[r, "\gamma_2"] & \dots \arrow[ull, hook, "\dots"']
\end{tikzcd}$$

As a final note on the subject, if you have encountered any connected sums $A \cup_f B$ before, you may notice that conventionally, these spaces were constructed by taking quotients $x \sim f(x)$ over disjoint unions $A \sqcup B$. Notice that this process is just taking coproducts and coequalizers. This is not a coincidence, and we will discuss this relationship between coproduct/coequalizers and colimits.