A brief introduction to category theory

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Outline

1. Why category theory was developed
2. The (developing) role of category theory
3. A showcase of what's to be expected
   - Diagram chases and abstract nonsense
   - The notion of duality
   - Structure as diagrams

Algebraic Topology

- Algebraic Topology relies on the computation of groups (or generally, algebraic invariants) of topological spaces – fundamental groups and (co)homology groups.
- At the point of category theory being developed, the subject of algebraic topology had already developed in a multitude of ways.[1]
- What is known to be category theory arises from an investigation of "natural equivalences".[1]

[1] There had already been multiple homotopy theories at this point, with attempts to unify it. Check *Fifty years of Homotopy Theory* by Whitehead for more details: `https://www.ams.org/journals/bull/1983-08-01/S0273-0979-1983-15072-3/S0273-0979-1983-15072-3.pdf`

- Category theory begins to act as a language for homological algebra, or by a better name, "abstract nonsense".
- By Grothendieck's contributions, the subject begins to detach from algebraic topology and is also utilized in algebraic geometry.[2]
- Also, the development of adjunctions begins to deepen category theory. Such a development births concepts such as limits and colimits[2], which benefit homotopy theory with the characterization of universal constructions.

[2] This is not by mistake. Theorems such as RAPL [3] show that there is a deeper connection between adjunctions and (co)limits than at first glance.

Major developments in how categories are used

Following the progress in category theory itself, the applications of categories to other areas of mathematics have became prevalent. Although always in the background, the use cases for category theory had become somewhat detached from purely topology – they were being used in various subjects, like algebraic geometry, logic and universal algebra to name a few.

Some of the progress in logic, achieved by Lawvere are[2]:

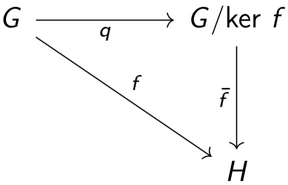
i. Axiomatizing the category of sets,
ii. Formulating completeness theorems for logical systems by categorical methods,
iii. Representing quantifiers as adjoint functors.

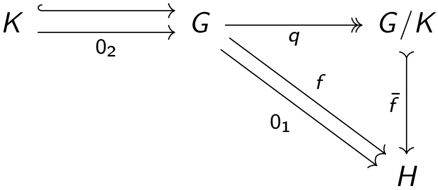
Diagrams

Diagrams often help simplify notions and constructions. Consider an abelian group G. Then, as a consequence of the first isomorphism theorem for groups, we have that

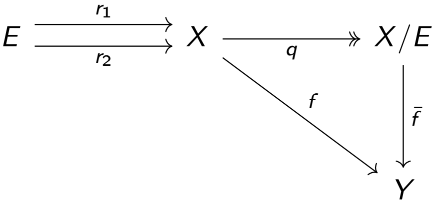
- For every (normal) subgroup $K \trianglelefteq G$, we have a group homomorphism $q : G \to S$ such that $\ker q = K$, and
- The kernel of every group homomorphism $f : G \to H$ is normal.

These properties are conventionally described by the diagram on the left, but can also be captured by this diagram (in Ab):





Similarly, for an equivalence relation E (with the discrete topology) over a space X gives the quotient space $q : X \to X/E$, the relationship of which can be shown by the diagram



where f is any continuous function with $f(r_1(e)) = f(r_2(e))$, which just means that f is constant under the same equivalence classes. If you recall any of the two definitions of a quotient space, you'll see why this condition is necessary; this way, for any continuous map keeping the equivalence classes the same, it actually passes through X/E and this is its **universal property**.

Duality

Example

Let $\Bbbk$ be a field, and let V be a vector space over $\Bbbk$. The dual space V^* of V is defined to be the following vector space over $\Bbbk$:

$$\text{Hom}(V, \Bbbk) = \{L : V \to \Bbbk \mid L \text{ is a } \Bbbk\text{-linear transformation.}\}$$

Then, for any $\Bbbk$-linear transformation $T : V \to W$, we have the induced "dual transformation"

$$T^* := \text{Hom}(T, \Bbbk) : W^* \to V^*$$
$$T^*(L : W \to \Bbbk) = L \circ T : V \to W \to \Bbbk$$

For a group G, suppose X is a (left) G-set. The action of each element $g \in G$ over X is a special function $g \cdot - : X \to X$. Now, the dual of this set with the group action is again a set X, but this time with a right G-action, where the compositionality of the group action is reversed:

$$\begin{aligned} \text{left action:} \quad & (gh) \cdot x = g \cdot (h \cdot x) \\ \text{right action:} \quad & (gh) \cdot_{\text{op}} x = h \cdot_{\text{op}} (g \cdot_{\text{op}} x) \\ \text{alternatively written:} \quad & x \cdot_{\text{op}} (gh) = (x \cdot_{\text{op}} g) \cdot_{\text{op}} h \end{aligned}$$

Diagrams internal to categories

Since there is support for diagrams, one can talk about structures:

Definition

A monoid is a set M, together with a binary action $\varepsilon : M \times M \to M$ and a dedicated element $e \in M$ such that

1. $\forall x \in M, \; ex = x = xe$
2. $\forall x, y, z \in M, \; (xy)z = x(yz)$.

We can rephrase this entirely diagrammatically as:

Definition

A monoid is a set M, together with a binary action $\varepsilon : M \times M \to M$ and a dedicated element $\eta : 1 \to M$ such that the following diagrams commute:

$$
\begin{array}{ccc}
M & \xrightarrow{1_M \times \eta} & M^2 \\
{\scriptstyle \eta \times 1_M}\downarrow & \searrow^{1_M} & \downarrow{\scriptstyle \varepsilon} \\
M^2 & \xrightarrow[\varepsilon]{} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M^3 & \xrightarrow{1_M \times \varepsilon} & M^2 \\
{\scriptstyle \varepsilon \times 1_M}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
M^2 & \xrightarrow[\varepsilon]{} & M
\end{array}
$$

where 1 is the monoid with one element. (*Note that* M^k just denotes the cartesian product k times – do not think of it as anything else.)

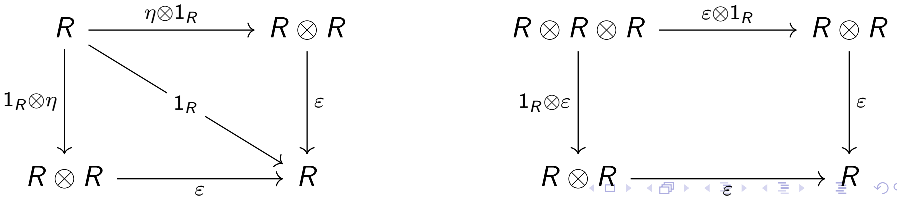
The central idea: **for an object in a category C, if it satisfies the diagrammatic monoid requirements, it is a C-monoid.**

Example

If M is a topological space with continuous maps $\eta : * \to M$ and $\varepsilon : M \times M \to M$ satisfying the monoid axioms, then it is a **topological monoid**.

Example

If R is an abelian group that also has maps $\eta : \mathbb{Z} \to R$ and $\varepsilon : R \otimes R \to R$ such that the following diagrams commute, then it is an abelian group monoid, or more commonly, a **unital ring!**



Limit and colimit objects

Example

Let X be a topological space, and $A \subset X$ any subset. Keep in mind that the closure of A in X is the smallest closed set $\bar{A}$ such that if any closed set B contains A, then it also contains $\bar{A}$.
Consider the partially ordered set $(\mathfrak{C}_A, \subset)$ with points as closed sets of X containing A, ordered under subset inclusion. Then the closure of A is an example of a limit – more specifically, of a diagram indexed by $(\mathfrak{C}_A, \subset)$.

Example

Similarly, considering the real numbers, given a non-empty subset $K \subset \mathbb{R}$ bounded above, the supremum $\underline{\sup K}$ is an example of a colimit object, one of a diagram indexed by the poset $(K, \leq)$ of elements of K ordered by the usual number order $\leq$.

Adjunctions

Here are two examples for a similar sort of adjunction:

Example

Let A, B, C be sets, and let C^B denote the set of all functions $B \to C$. Then, notice that for any function $f : A \times B \to C$, if we fix $a \in A$, we obtain a new function $f(a, -) : B \to C$. Similarly, if we have a function $g : A \to C^B$, then we can obtain a new function $\bar{g} : A \times B \to C$ by defining $\bar{g}(a, b) := g(a)(b)$.[a] In an expression, this just means we have the following bijection:

$$\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C^B)$$

[a] In computer science and mostly in type theory, this sort of destructuring is often called *currying*, named after Haskell Curry. In type systems, it is the case that $A \times B \to C$ and $A \to B \to C$ are considered to be the same thing. Refer to [4] for more details.

Example

Suppose U, W, V are vector spaces over some fixed field $\Bbbk$. Note that the previous example works functionally the same:

$$\text{Linear}(A \otimes B, C) \cong \text{Linear}(A, C^B)$$

Please note that C^B is the vector space of linear transformations $B \to C$, turned into a vector space pointwise: $(cT + G)(v) = cT(v) + G(v)$. The symbol $\otimes$ denotes the tensor product $(\otimes_{\Bbbk})$.

Citations

[1] Samuel Eilenberg **and** Saunders MacLane. "General theory of natural equivalences". **in** *Transactions of the American Mathematical Society*: 58.2 (1945), **page** 231. DOI: 10.2307/1990284.

[2] Jean-Pierre Marquis. *Category theory*. **august** 2019. URL: https://plato.stanford.edu/entries/category-theory/#BrieHistSket.

[3] Emily Riehl. *Category theory in context*. Dover Publications, 2017.

[4] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.